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REGULATION OF SPLIT LINEAR SYSTEMS OVER RINGS: COEFFICIENT-ASSI--ETC(U)
FEB 80 E EMRE, P P KHARGONEKAR DAAG29-80-C-0050

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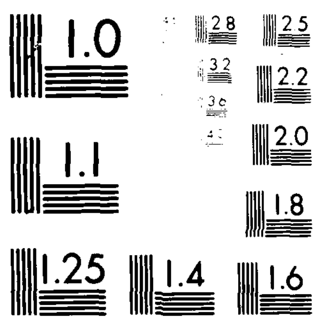

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REGULATION OF SPLIT LINEAR SYSTEMS OVER RINGS:
COEFFICIENT-ASSIGNMENT AND OBSERVERS

E. Emre and P.P. Khargonekar

Reprinted from IEEE Transactions on Automatic Control, Vol. AC-27, No. 1, February 1982
0018-9286/82/0200-0104$00.75

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 15077.17-M	2. GOVT ACCESSION NO. N/A AD-A112 145	3. RECIPIENT'S CATALOG NUMBER N/A
4. TITLE (and Subtitle) Regulation of Split Linear Systems Over Rings: Coefficient-Assignment and Observers		5. TYPE OF REPORT & PERIOD COVERED Reprint
		6. PERFORMING ORG. REPORT NUMBER N/A
7. AUTHOR(s) E. Emre, P. P. Khargonekar		8. CONTRACT OR GRANT NUMBER(s) DAAG29 80 C 0050
9. PERFORMING ORGANIZATION NAME AND ADDRESS University of Florida Gainesville, FL 32611		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS N/A
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P. O. Box 12211 Research Triangle Park, NC 27709		12. REPORT DATE Feb 82
		13. NUMBER OF PAGES 10
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Submitted for announcement only.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

Regulation of Split Linear Systems Over Rings: Coefficient-Assignment and Observers

E. EMRE AND P. P. KHARGONEKAR

***Abstract*—A theory of regulators is developed for finite-free split linear systems over a commutative ring K. This is achieved by developing a theory of coefficient-assignment and observers. It is shown that the problem of coefficient-assignment can be solved for reachable systems by using dynamic state feedback. For strongly observable systems, it is shown that observers with arbitrary dynamics can be constructed. Internal stability of observer and state feedback configuration is considered explicitly. Some examples are given to illustrate these techniques.**

Manuscript received February 22, 1980; revised August 28, 1980 and December 9, 1980. Paper recommended by B. Francis, Past Chairman of the Linear Systems Committee. This work was supported in part by the U.S. Army under Research Grant DAA29-80-C-0050 and in part by the U.S. Air Force under Grant AFOSR 76-3034 Mod-C.

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I. Introduction

Linear systems over commutative rings have been considered by several authors (see [21], [22], [5], [14], [23], and the references therein). For applications and a survey of results on systems over rings, the reader is referred to the survey papers by Kamen [15] and Sontag [23]. Among the classes of systems that can be formulated in this framework are, for example, delay differential systems, systems whose coefficients depend on parameters, systems whose coefficients are integers, multidimensional systems, and systems described by discretized partial differential equations.

By a *finite-free linear system over a commutative ring* K, is meant a finitely generated free K-module X of rank n, and K-linear maps

$$F: X \to X,$$
$$G: K^m \to X,$$
$$H: X \to K^p.$$

Without loss of generality, one can assume that F, G, H are $n \times n$, $n \times m$, and $p \times n$ matrices over K. We will denote such a system as $\Sigma = (F, G, H)$ and will associate with it the abstract discrete-time system

$$x(t+1) = Fx(t) + Gu(t)$$
$$y(t) = Hx(t).$$

This is only a conventional way of representing such systems. However, the constructions obtained based on such representations, in each case of interest, are valid for the actual interpretations of the system Σ through certain transforms. The following examples illustrate how this is done.

Example 1.1 — A Delay-Differential System: Consider the system

$$\dot{x}_1(t) = 5x_1(t-2) + x_2(t) + u_1(t)$$
$$\dot{x}_2(t) = x_1(t) + x_2(t-\pi) + u_2(t-2)$$
$$y_1(t) = x_1(t) - x_2(t-\pi)$$
$$y_2(t) = x_2(t).$$

Defining the delay operators σ_1, σ_2 by

$$\sigma_1(x)(t) := x(t-2)$$
$$\sigma_2(x)(t) := x(t-\pi),$$

we can represent the above system as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 5\sigma_1 & 1 \\ 1 & \sigma_2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & \sigma_1 \end{bmatrix}\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & -\sigma_2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

Hence, a representation for this system is

$$\Sigma := \left(\begin{bmatrix} 5\sigma_1 & 1 \\ \sigma_1 & \sigma_2 + 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & \sigma_1 \end{bmatrix}, \begin{bmatrix} 1, & -\sigma_2 \\ 0 & 1 \end{bmatrix}\right)$$

which is a system over the polynomial ring $K := \mathbf{R}[\sigma_1, \sigma_2]$ (where $\mathbf{R} :=$ real numbers). It carries all the information about the original system.

Example 1.2 — A System with Parameters: Suppose that we have a system $\Sigma = (F, G, H)$ with parameter uncertainty. One could represent this uncertainty by assuming that the entries of F, G, H are functions of a finite number of parameters, say $\alpha_1, \cdots, \alpha_n$. These functions, for example, could be polynomial functions in which case Σ would be a system over the polynomial ring $\mathbf{R}[\alpha_1, \cdots, \alpha_n]$. An example is

$$F = \begin{bmatrix} 1 & 1 \\ \alpha_2 & \alpha_1^2\alpha_1 \end{bmatrix}$$

$$G = \begin{bmatrix} 0 & \alpha_1^2 \\ 1 & \alpha_2 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & \alpha_1^2 + \alpha_2^2 \\ \alpha_1 + 1 & \alpha_2 \end{bmatrix}.$$

In such cases one would want to design a regulator which would work for every known value of α_1 and α_2 by adjustment of its parameters. This can be done by designing a regulator over the ring $\mathbf{R}[\alpha_1, \alpha_2]$.

Further examples can be found in [23], [15].

A finite-free system $\Sigma = (F, G, H)$ of dimension n over a commutative ring K is *reachable* iff

$$K^n = \operatorname{Im} G + \operatorname{Im} FG + \cdots + \operatorname{Im} F^{n-1}G,$$

i.e., the columns of the reachability matrix

$$\Omega_R = [G : FG : \cdots : F^{n-1}G]$$

generate the module K^n. In such a case we also say that (F, G) is reachable.

Σ is *observable* iff the columns of the observability matrix

$$\Omega_0 = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix}$$

are K-linearly independent.

This is equivalent to the condition that $x \in K^n$, and $\Omega_0 x = 0$ implies $x = 0$.

Σ is *strongly observable* (respectively, *weakly reachable*) iff (F', H') (transposes of F and H) (respectively, (G', F')) is a reachable (respectively, observable) pair.

We should note that in general strong observability implies observability but not vice versa. However, in the case $K =$ a field these two notions coincide.

In general, a *strictly proper rational transfer matrix* over a commutative ring K is defined as a formal power series

$$Z := A_1 z^{-1} + A_2 z^{-2} + \cdots$$

where A_i, $i = 1, 2, \cdots$, is a $p \times m$ matrix over K, and the matrices A_i's satisfy a *recurrence relation*

$$A_{i+n} = \sum_{j=0}^{n-1} \alpha_j A_{i+j}; \qquad i = 1, 2, \cdots$$

for some $\alpha_j \in K$, and some integer n.

Such a transfer matrix always has a realization (F, G, H) such that

$$Z = H(zI - F)^{-1}G.$$

For general results on realization theory the reader is referred to [21]–[23], [15] and the references therein. For results of major importance to our paper the reader is referred to Section II.

The problem of finding an integer n_c and constructing feedback compensators for a system (F, G, I) so that the closed-loop transfer matrix has a finite-free realization of dimension n_c, (F_c, G_c, H_c), where the characteristic polynomial of F_c can be arbitrarily assigned is called the problem of *coefficient-assignment*. If we only require that the characteristic polynomial of F_c has n_c roots $z_0, \cdots, z_{n_c}$ with $z_i \in K$ which can be chosen arbitrarily, then this problem is called the problem of *pole-placement*. In general coefficient-assignment implies pole-placement but not vice versa. In the case $K =$ a field the two imply each other, and are well known to be equivalent to the reachability of (F, G) and have well known constructive solutions (see [25], [20], [1], [13], [6], [7], and the references therein).

However, the case where $K \neq$ a field is more difficult and only partial results have been obtained previously. The main approach has been to extend the results for the field case using constant state-feedback. Morse [19] has shown that if K is a principal ideal domain and (F, G) is reachable, then pole assignment is possible by nondynamic state-feedback. Sontag [23] has proved that for semilocal rings coefficient-assignment by nondynamic feedback is equivalent to the reachability of (F, G).

Later Ching and Wyman [4] proved that Noetherian full quotient rings are semilocal rings and thus the results of Sontag [23] are applicable to this case. In general, Sontag [23] has proved that the reachability of (F, G) is necessary for both pole-placement and coefficient-assignment by nondynamic feedback for any ring K.

Let P_s be a multiplicatively closed subset of monic polynomials of the polynomial ring $K[z]$, called *stable polynomials*. Let (F, G) be a given pair. We say that (F, G) is *regulated* (or *regulation* is achieved) iff there exists a finite-free system $\Sigma_1 = (F_1, G_1, H_1, J_1)$ such that with dynamic feedback using Σ_1, the characteristic polynomial of the augmented closed-loop system is in P_s. (This latter property is usually referred to as *internal stability* as well.)

For principal ideal domains and semilocal rings and Noetherian full quotient rings, the results of [19], [23], [4], respectively, can be used to solve the regulation problem. Also, Byrnes [3] has given a stabilization result that can be used for regulation of systems over certain Frechet algebras.

However, for more general rings, there do not exist any coefficient-assignment and/or pole-placement results that can be used for regulation problem for a reachable pair (F, G). In Section II of this paper we present for the first time a coefficient-assignment result by dynamic feedback which can be used to solve the regulation problem for finite-free reachable pairs (F, G) defined over an arbitrary commutative ring K. Our result is based on an approach developed in [7], valid for an arbitrary commutative ring K, which also connects the problem to the reachability of (F, G) in a natural way.

The *general regulation problem* is to achieve regulation as defined above when the plant is described as $\Sigma = (F, G, H)$. If a finite-free pair (H, F) is strongly observable, one can build an observer (in the form of a deterministic Kalman filter) for the rings mentioned above where a result of coefficient-assignment and/or pole-placement by nondynamic feedback is available.

$\Sigma = (F, G, H)$ is called *split* iff it is reachable and strongly observable. For the motivation and details of the notion of split systems (see [23], [16]). For a finite-free split system Σ define over K where K is a principal ideal domain or a semilocal ring, one can solve the regulator problem by combining the results on observers for pair (H, F) and results on regulators for pairs (F, G), or for K = a certain Frechet algebra, results of [3] can be used to solve regulation problem. Recently, Hautus and Sontag [11] have obtained more general results on observers, also generalizing detectability ideas, for finitely generated algebras over fields. They have also considered the regulation problem for finite-free detactable and reachable systems over such rings, however their results cannot be used directly for regulation. In Section III of this paper, we give for the first time, a method to obtain an observer for a finite-free strongly observable system $\Sigma = (F, G, H)$ over K = an arbitrary commutative ring, based on the approach developed in [7]. This also connects this problem to the observability of Σ in a natural way. Further, our observers can be built in such a way that the coefficient of the characteristic polynomial can be chosen arbitrarily. Also, our results allow the observers to be built by solving linear equations over K, and are guaranteed to have a realization over K.

Further, we show how one can combine our results on observers and coefficient-assignment to solve the general regulation problem for a finite-free split linear system $\Sigma = (F, G, H)$ over an arbitrary commutative ring K.

One can also use our coefficient-assignment result and the detectability result of [11] to solve the regulator problem for the case of finite-free detectable and reachable systems $\Sigma = (F, G, H)$ over finitely generated algebras over fields.

In Section IV, we give some examples to illustrate the applications of our results. We show that even if $\Sigma = (F, G, H)$ is not reachable and/or strongly observable, one can still approach and may be able to solve the problem using our method.

As for $\Sigma = (F, G, H)$ being split, over a commutative ring, every transfer matrix admits a free reachable or a free strongly observable realization. In general, reachability is known to be necessary for coefficient-assignment by state feedback [23]. For the existence of observers with arbitrary characteristic polynomial for a weakly reachable system, strong observability (as it will be seen in Section III) is also necessary. We show in Sections II and III that for free systems these are also sufficient. Furthermore, in the case of systems over $\mathbb{R}[x_1, \cdots, x_r]$, $\Sigma = (F, G, H)$ being reachable (strongly observable) is generic if and only if the number of inputs (outputs) exceeds r (see [18]). Hence, in such cases, the assumption of splitness is not very restrictive.

The choice of stable polynomials P_s, of course, depends upon the application under consideration. For most of the applications, stability can be inferred from the characteristic polynomial. For example, for delay-differential systems, two-dimensional systems, systems with parameters, one can infer stability by examining the location of roots of the characteristic polynomial. Throughout this paper, we assume that P_s is given and derive results for the given P_s. The only technical restriction on P_s is that it be a multiplicative set.

II. Coefficient-Assignment for Linear Systems Over Commutative Rings

In this section we will prove our main "coefficient-assignment" theorem, for a reachable system (F, G, I) over a commutative ring K, which also guarantees regulation. Similar results will be used in Section III to construct observers for finite-free strongly observable systems over K. Then these results together will be shown to provide a method for design of compensators for the solution to the regulation problem for finite-free split systems over K.

Throughout the paper K will denote a commutative ring with identity. Let $K[z]$ denote the ring of polynomials in the indeterminate z with coefficients in K. Let $K(z)$ denote the ring of formal power series in z^{-1} with coefficients in K. If p and q are in $K[z]$ and q is monic then p/q is identified with the formal power series obtained by the formal division of p by q. Define the K-linear map π as

$$\pi: K(z) \to K(z)/K[z]: \sum_{i=-n}^{\infty} a_i z^{-i} \to \sum_{i=1}^{\infty} a_i z^{-i}.$$

A formal power series a is said to be *strictly proper* if $\pi(a)$ is a, and is said to be proper if $\pi(a) - a$ is an element of K. For a set S and positive integers p and m, S^p, and $S^{p \times m}$ denote the set of p-column vectors and the set of $p \times m$ matrices with entries in S. The map π is extended to $K^p(z)$ in the natural way.

Let Q be a $p \times p$ nonsingular polynomial matrix such that determinant of Q, $|Q|$, is a monic polynomial. Define the K-module K_Q as

$$K_Q := \{x \text{ in } K^p[z] : Q^{-1}x \text{ is strictly proper}\}.$$

The K-linear map π_Q is defined as

$$\pi_Q : K^p[z] \to K_Q : x \mapsto Q\pi(Q^{-1}x).$$

Let Φ be in $K^{p \times r}[z]$ with φ_i as its ith column. Define $\pi_Q(\Phi)$ to be the $p \times r$ polynomial matrix whose ith column is $\pi_Q(\varphi_i)$. It is easy to see that for a given $p \times r$ polynomial matrix Φ, there exists a unique $p \times r$ polynomial matrix Q_1 such that

$$\Phi = QQ_1 + \pi_Q(\Phi).$$

For a K-linear map f, let $\operatorname{im} f$ denote the image of f and let $\ker f$ denote the kernel of f. For a matrix B whose columns are elements of a K-module M, let $Sp_K B$ denote the submodule of M generated by the columns of B. For a polynomial matrix P, let $\delta_{ci}(P)$ denote the degree of ith column of P. A nonsingular polynomial matrix Q is said to be *row* (*column*) *proper*, iff the highest row (column) degree coefficient matrix is invertible over the ring K. Note that if Q is row proper with row degrees $n_1, n_2, \cdots, n_p$, then K_Q is a free K-module with a basis given by the columns of the $p \times n$ polynomial matrix

$$S := \operatorname{diag}(V_1, V_2, \cdots, V_p)$$

where

$$V_i := (z^{n_i - 1}, z^{n_i - 2}, \cdots, 1)$$

and

$$n := n_1 + n_2 + \cdots + n_p.$$

Let Z denote a $p \times m$ strictly proper transfer matrix over the ring K. If we express Z as

$$Z = \sum_{i=1}^{\infty} A_i z^{-i},$$

then the sequence $\{A_i\}_{i=1}^{\infty}$ is the sequence of impulse response matrices of the i/o map associated with Z. Let P, Q, and R be $p \times r$, $r \times r$, and $r \times m$ polynomial matrices with determinant of Q a monic polynomial such that

$$Z = PQ^{-1}R.$$

The following lemma presents a natural realization of Z with K_Q as the state-module. Since $|Q|$ is assumed to be monic, it is not difficult to see that K_Q is a finitely generated K-module. The realization given in the following lemma is called the Q-realization of Z. This result was proved in [10] for K a field. But since π_Q is well defined, it is easily seen that the result holds for our case as well.

Lemma 2.1 [10]: Let $\Sigma_Q = (F_Q, G_Q, H_Q)$ *be defined as follows:*

$$G_Q: K^m \to K_Q: \ u \mapsto \pi_Q(Ru)$$

$$F_Q: K_Q \to K_Q: \ x \mapsto \pi_Q(zx),$$

and

$$H_Q: K_Q \to K^p: \ x \mapsto (PQ^{-1}x)_{-1}$$

where for a in $K^p(z)$, $(a)_{-1}$ denotes the coefficients of z^{-1}. Then Σ_Q with K_Q as the state-module is a realization of Z.

For further details of the Q-realization, the reader is referred to [10], [8], and the references given there.

Let $\Sigma = (F, G, H)$ be a given finite-free split system over K. Let Z be its transfer matrix, i.e.,

$$Z := H(zI - F)^{-1}G.$$

We will assume, without loss of generality, that Z can be written as

$$Z = Q^{-1}R$$

where Q, R are polynomial matrices, Q is row proper, and Q, R satisfy the Bezout condition, i.e., there exist polynomial matrices X, Y such that

$$QX + RY = I.$$

If K = a field such a representation is known to be possible for a reachable and observable system Σ. But this is generally not possible for $K \neq$ a field. However, if Σ is split using a full state observer (as described in full detail in Section III) one can obtain a system whose transfer matrix is

$$Z_f := (zI - F)^{-1}G$$

which clearly is of the above form. Further, if this is done, then regulation of the composite system (original system Σ and the observer) is sufficient for regulation of Σ, as it will be shown in Remark (3.9).

Now consider a dynamic feedback system $\Sigma_1 = (F_1, G_1, H_1, J_1)$ over K, whose transfer matrix Z_1 can be written as

$$Z_1 = P_1 Q_1^{-1}$$

where P_1, Q_1 are polynomial matrices with Q_1 column proper. (Later in this section we will show how to obtain such P_1 and Q_1 for regulation.) Then if we write the closed-loop state equations where the state is taken to be the external direct sum of the states of Σ, the observer and Σ_1, it can be shown that [see Remark 3.9] the closed-loop system has the character polynomial α_c, where

$$\alpha_c = |\Phi||Q_0|.$$

Here

$$Q_0^{-1}[TG \ : \ R_2]$$

will be the transfer matrix of the observer;

$$\Phi = (zI - F)Q_1 + GP_1$$

and

$$Q_0 = T(zI - F) + R_2 H.$$

In this section, we will show how one can obtain polynomial matrices Φ, P_1, Q_1 such that $|\Phi|$ can be arbitrarily chosen, Q_1 is column proper and $P_1 Q_1^{-1}$ is a proper rational matrix. In Section III, we will show how one can obtain suitable polynomial matrices Q_0, T, R_2.

We should note that dynamic state feedback corresponds to adding a number of integrators to the system (F, G, I) and then applying state feedback to the resulting system. (See, for example, [1].) Hence, the results that we present in this section can also be interpreted as a method to determine a sufficient number of integrators, and the final state feedback matrix for regulation.

Our results in this section are based on the following theorem proved in [7, Theorem 3.1].

For each positive integer i, define a K-submodule W_i of K_Q as

$$W_i := \operatorname{Im} G_Q + \operatorname{Im} F_Q G_Q + \cdots + \operatorname{Im} F_Q^i G_Q.$$

In the following for a $p \times p$ polynomial matrix Φ and integers $\{\alpha_i\}_{i=1}^{p}$, $\{\beta_i\}_{i=1}^{p}$,

$$\lim_{z \to \infty} \{\operatorname{diag}(z^{-\alpha_i})\Phi \operatorname{diag}(z^{-\beta_i})\}$$

exists and is equal to $\hat{T}$, where $\hat{T}$ is a matrix over K, means that

$$Z_t := \operatorname{diag}(z^{-\alpha_i})\Phi \operatorname{diag}(z^{-\beta_i})$$

is proper and its constant term is $\hat{T}$.

Lemma 2.2 [7, Theorem 3.1]: Let Φ be a given $p \times p$ polynomial matrix. Let Q be a $p \times p$ nonsingular matrix with I_p as the highest row degree coefficient matrix and with row degrees $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_p$. Let R be given $p \times m$ polynomial matrix such that $Q^{-1}R$ is strictly proper. Let P_1 be a given $m \times p$ polynomial matrix. Then there exists a column proper nonsingular polynomial matrix Q_1 such that $P_1 Q_1^{-1}$ is proper, and

$$QQ_1 + RP_1 = \Phi \tag{2.3}$$

if and only if there exist integers $\gamma_1, \gamma_2, \cdots, \gamma_p \geq 0$ such that

$$\text{i)} \quad \lim_{z \to \infty} \left\{ \begin{bmatrix} z^{-\mu_1} & & 0 \\ & \ddots & \\ 0 & & z^{-\mu_p} \end{bmatrix} \Phi \begin{bmatrix} z^{-\gamma_1} & & 0 \\ & \ddots & \\ 0 & & z^{-\gamma_p} \end{bmatrix} \right\} = A$$

exists and A is invertible over the ring K, and

$$\text{ii)} \quad \gamma_i \geq \delta_{ci}(P_1) \geq r_i - 1, \qquad i = 1, 2, \cdots, p,$$

where r_i is the least integer j for which the ith column of $\pi_Q(\Phi)$, φ_i, is in W_{j-1} and P_1 is a polynomial matrix whose ith column is an input which drives Σ_Q from zero-state to φ_i. Further if i) and ii) hold, then A is the highest degree column coefficient matrix of Q_1, and γ_i is the ith column degree of Q_1.

The following theorem is an immediate consequence of Lemma 2.2.

Theorem 2.4: Let Q be a $p \times p$ nonsingular polynomial matrix with I_p as the highest degree row coefficient matrix. Let R be a $p \times m$ polynomial matrix such that the Q-realization Σ_Q of $Q^{-1}R$ is reachable. Let ν be the smallest integer such that

$$K_Q = \operatorname{Im} G_Q + \operatorname{Im} F_Q G_Q + \cdots + \operatorname{Im} F_Q^{\nu-1} G_Q.$$

Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_p$ be the row degrees of Q. Let $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_p$ be a given set of positive integers such that

$$\rho_i \geq \mu_i + \nu - 1.$$

Let β be a given monic polynomial of degree

$$n := \rho_1 + \rho_2 + \cdots + \rho_p.$$

Then there exists a $p \times p$ polynomial matrix Q_1 with column degrees $\rho_1 - \mu_1, \rho_2 - \mu_2, \cdots, \rho_p - \mu_p$ and I_p as the highest degree column coefficient matrix, and there exists an $m \times p$ polynomial P_1 such that $P_1Q_1^{-1}$ is a proper transfer matrix and β is the determinant of $QQ_1 + RP_1$.

Proof: By hypothesis, we have

$$\rho_1 \geq \rho_2 \geq \cdots \rho_p.$$

Following the procedure given in [24, Section 7.3], [12, Section 7.2] define the polynomials $\beta_1, \beta_2, \cdots, \beta_p$ by

$$\beta = z^n + \beta_1 z^{n-\rho_1} + \beta_2 z^{n-\rho_1-\rho_2} + \cdots + \beta_p$$

where β_i is a polynomial of degree less than ρ_i. Now if the $p \times p$ polynomial matrix Φ is defined as

$$\Phi := \begin{bmatrix} z^{\rho_1} + \beta_1 & \beta_2 & \cdots & \beta_p \\ -1 & z^{\rho_2} & & \\ \vdots & & \ddots & \vdots \\ 0 & & -1 & z^{\rho_p} \end{bmatrix},$$

then $|\Phi|$ is β. Let $\gamma_i := \rho_i - \mu_i$, $i = 1, 2, \cdots, p$. Then

$$\lim_{z \to \infty} \left\{ \begin{bmatrix} z^{-\mu_1} & \cdots & & 0 \\ & z^{-\mu_2} & & \\ \vdots & & \ddots & \\ 0 & & & z^{-\mu_p} \end{bmatrix} \Phi \begin{bmatrix} z^{-\gamma_1} & \cdots & & 0 \\ & z^{-\gamma_2} & & \\ \vdots & & \ddots & \\ 0 & & & z^{-\gamma_p} \end{bmatrix} \right\} = I_p.$$

Now the conditions of Lemma 2.2 are satisfied, hence there exist polynomial matrices Q_1 and P_1 with the required properties and

$$QQ_1 + RP_1 = \Phi.$$

This completes the proof of the theorem. □

We also have the following.

Theorem 2.5: Let F in $K^{n \times n}$ and G in $K^{n \times m}$ be a reachable pair. Let ν be the smallest integer such that

$$K^n = \operatorname{Im} G + \operatorname{Im} FG + \cdots + \operatorname{Im} F^{\nu-1}G.$$

Let $\rho_1, \rho_2, \cdots, \rho_n$ be a given set of integers such that each $\rho_i \geq \nu$, and let β be a given monic polynomial of degree

$$\rho := \rho_1 + \rho_2 + \cdots + \rho_n.$$

Then there exists a nonsingular polynomial matrix Q_1 with highest column degree coefficient matrix I_n and with $\rho_i - 1$ as ith column degree, and an $m \times n$ polynomial matrix P_1 such that $P_1Q_1^{-1}$ is proper and

$$|(zI - F)Q_1 + GP_1| = \beta.$$

Proof: If we choose

$$Q = (zI - F), \quad R = G$$

then the Q-realization of $Q^{-1}R$ is (I, F, G) with K^n as the state module. Now we have

$$\mu_1 = \mu_2 = \cdots = \mu_p = 1.$$

The result follows immediately from Theorem 2.4. □

Remark 2.6: Let Q and R be as in Theorem 2.4 and let Σ_Q be reachable. It is well known (see, for example, [7]) that if $Z_1 = -P_1Q_1^{-1}$ is chosen as a feedback compensator, then the closed-loop transfer matrix Z_c can be written as

$$Z_c = Q_1[QQ_1 + RP_1]^{-1}R$$

which is the same as

$$Z_c = Q_1\Phi^{-1}R.$$

Note that Φ' is row proper. One can see by obtaining a Φ' realization of Z_c' as in Lemma 2.1, and choosing S as a basis matrix for $K_{\Phi'}$, that F_Φ has a matrix representation which is in companion form with the characteristic polynomial

$$|\Phi| = \beta.$$

As this realization of Z_c' is free, one obtains a realization of Z_c by matrix transpositions. Hence, Theorem 2.5 not only provides us with a coefficient-assignment result, but it also gives a realization of Z_c with a F_Φ in companion form.

Remark 2.7: Theorem 2.5 is a coefficient-assignment result for reachable pairs (F, G) by dynamic state-feedback which guarantees the internal stability (see Remark 3.8 for details of this aspect) of the closed-loop system, thus providing a solution to the regulator problem. All of the previous work except for Hautus and Sontag [11] has been concerned with coefficient-assignment or stabilization by constant state feedback. In [11] a special type of pole-placement result for reachable pairs (F, G) by dynamic state-feedback is given but the internal stability of the closed-loop system is not considered.

Theorem 2.5 is the first coefficient-assignment result for reachable pairs (F, G) and further it also guarantees regulation. It will be seen in the following section that one can build (at least) a full state observer for finite-free strongly observable systems over arbitrary commutative rings. These results together provide a solution to coefficient-assignment and regulation problems (as shown in Remark 3.8 in detail) for finite-free split linear systems defined over arbitrary commutative rings.

Remark 2.8: Concerning the construction of compensators, note that Q_1' is row proper. Therefore, $K_{Q_1'}$ is finite and free. Hence, using Lemma 2.1, we can obtain matrix representations of the Q_1'-realization of $Q_1'^{-1}P_1'$, and then transpose them to obtain a realization of $Z_1 = P_1Q_1^{-1}$.

III. Observers for Linear Systems Over Commutative Rings

In this section, we will show how one can build observers for finite-free strongly observable linear systems over a commutative ring K. Throughout this section, without loss of generality, we will assume that all the systems under consideration have dynamic interpretations as discrete-time systems over K.

Recall that $K[z]$ has a multiplicatively closed subset of monic polynomials P_s called the set of *stable polynomials*. A rational function p/q where $p, q \in K[z]$ will be said to be stable if $q \in P_s$.

Let $\Sigma = (F, G, H)$ be a linear system over K with the dynamic interpretation

$$x(t+1) = Fx(t) + Gu(t),$$
$$y(t) = Hx(t).$$

We assume that F, G, H are $n \times n$, $n \times m$, $p \times n$ matrices over K and K^n is the state module. K^m and K^p are input and output value modules. We define the transfer matrices Z and Z_s as

$$Z = H(zI - F)^{-1}G$$
$$Z_s = (zI - F)^{-1}G.$$

Let L be an $l \times n$ matrix over K. Let $\Sigma_L = (F_o, [G_{o1} : G_{o2}], H_o, [0 : J_o])$ where $F_o, G_{o1}, G_{o2}, H_o, J_o$ are $r \times r$, $r \times m$, $r \times p$, $l \times r$, and $l \times p$ matrices over K be a system with state module K^r, input value module K^{m+p}, output value module K^l. The dynamic interpretation of Σ_L is

$$\hat{x}(t+1) = F_o\hat{x}(t) + G_{o1}u(t) + G_{o2}y(t)$$
$$\hat{y}(t) = H_o\hat{x}(t) + J_oy(t).$$

We call Σ_L an *L-observer* for Σ iff for every initial state x_o of Σ and every initial state $\hat{x}_o$ of Σ_L and for every input u each of the transfer matrices from $u, x_o, \hat{x}_o$ to

$$e := \hat{y} - Lx$$

is stable.

We should note here that the degree of stability of the transfer matrices

in the above definition is also very important. Because this determines the rate of convergence (for the rings where convergence is well defined), and as it will be shown in Remark 3.8 the characteristic polynomial of F_o affects the regulation of the closed-loop system. Therefore, it is more desirable to be able to assign the coefficients of $|zI - F_o|$ arbitrarily. It will be seen that this can be achieved with our approach.

We should note here that x_o, $\hat{x}_o$ do not have to be in K^n and K^r. In general, they may be in some other sets. But as we will see, our results are independent of the sets that x_o, $\hat{x}_o$ may belong to. The reader is referred to Hautus and Sontag [11] for a discussion of this aspect of the problem.

To establish our main result, first we need the following two lemmas.

Lemma 3.1: *Let R_1, R_2, T, Q_o be $l \times m$, $l \times p$, $l \times n$, and $l \times l$ polynomial matrices over K such that*

$$(zI - F')T' + H'R_2' = L'Q_o' \tag{3.1a}$$

$$R_1 = TG. \tag{3.1b}$$

Then

$$Q_o^{-1}[R_1 \,:\, R_2]\begin{bmatrix} I_m \\ z \end{bmatrix} = LZ_s. \tag{3.2}$$

Conversely, if (F, G) is weakly reachable and (3.2) is satisfied for some polynomial matrices Q_c, R_1, R_2, then there exists an $l \times n$ polynomial matrix T such that (3.1a) and (3.1b) are satisfied.

Proof: (3.1a) and (3.1b) imply that

$$T(zI - F) = Q_o L - R_2 H$$

and

$$TG = R_1$$

or

$$(Q_o L - R_2 H)(zI - F)^{-1}G = -R_1,$$

which yields (3.2).

Conversely, if (3.2) holds, we can rewrite it as

$$(R_2 H - Q_o L)(zI - F)^{-1}G = -R_1.$$

Let

$$V(z) := R_2 H - Q_o L = \sum_{j=0}^{q} V_j z^j.$$

Then, as R_1 is polynomial,

$$[V_o \,:\, \cdots \,:\, V_q]\begin{bmatrix} I \\ \vdots \\ F^q \end{bmatrix}[G \,:\, FG \,:\, \cdots] = 0.$$

As (F, G) is weakly reachable, we must have

$$[V_o \,:\, \cdots \,:\, V_q]\begin{bmatrix} I \\ \vdots \\ F^q \end{bmatrix} = 0.$$

But this shows that $V(z)$ is right divisible by $(zI - F)$, i.e., there exists an $l \times n$ polynomial matrix T such that

$$(R_2 H - Q_o L) = -T(zI - F)$$

and

$$R_1 = TG. \qquad \square$$

Lemma 3.3: *Let R_1, R_2, Q_o be as in Lemma 3.1 with $|Q_o| \in P_s$, and such that $Q_o^{-1}[R_1 \,:\, R_2]$ has a realization $\Sigma_L = (F_o, [G_{o1} \,:\, G_{o2}], H_o, [0 \,:\, J_o])$ with the state module K^r (for some integer $r \geq 0$), and $|zI - F_o| = |Q_o|$. Then Σ_L is an L-observer for Σ.*

Proof: Let u be a given input. Let x_o be an initial state for Σ and $\hat{x}_o$ be an initial state for Σ_l. Then one can write the formal z-transform $\hat{e}$ of e as

$$\hat{e} = Z_1 + Z_2 + Z_3 + Z_4 + Z_5$$

where

$$Z_1 = Q_o^{-1}[R_1 \,:\, R_2]\begin{bmatrix} I_m \\ Z \end{bmatrix}\hat{u},$$

$$Z_2 = Q_o^{-1}R_2 H(zI - F)^{-1}zx_o,$$

$$Z_3 = H_o(zI - F_o)^{-1}z\hat{x}_o,$$

$$Z_4 = -L(zI - F)^{-1}G\hat{u},$$

$$Z_5 = -L(zI - F)^{-1}zx_o.$$

But from Lemma 3.1 we have

$$Z_1 + Z_4 = 0.$$

Hence,

$$\hat{e} = Z_2 + Z_3 + Z_5.$$

But as $|zI - F_o| = |Q_o|$ is in P_s, if $Z_2 + Z_5$ depends on x_o and $\hat{x}_o$ through stable rational matrices, so does $\hat{e}$. Now we will prove that this is the case. Consider

$$z^{-1}(Z_2 + Z_5) = Q_o^{-1}(R_2 H - Q_o L)(zI - F)^{-1}x_o.$$

From (3.1a) we have

$$z^{-1}(Z_2 + Z_5) = -Q_o^{-1}Tx_o.$$

As $|Q_o|$ is in P_s, $\hat{e}$ depends on x_o, $\hat{x}_o$, and u through stable rational matrices. Hence, by definition of an L-observer, Σ_l is an L-observer for Σ. $\square$

Now we state our main theorem regarding L-observers.

Theorem 3.4: *Let (F', H') be reachable. Let μ be the first integer such that*

$$K^n = \operatorname{Im} H' + \cdots + \operatorname{Im} F'^{\mu-1}H',$$

(i.e., μ is the reachability index of (F', H').) Let $\gamma_1, \cdots, \gamma_l$ be integers such that $\gamma_i \geq \mu - 1$. Let α be a given monic polynomial of degree

$$\gamma = \sum_{i=1}^{l} \gamma_i.$$

Then, there exist polynomial matrices Q_o, R_1, R_2 satisfying (3.1a) such that Q_o is row proper with ith row degree γ_i, $|Q_o| = \alpha$, and such that $Q_o^{-1}[R_1 \,:\, R_2]$ has a finite realization

$$\Sigma_L := (F_o, H_o, [G_{o1} \,:\, G_{o2}], [0 \,:\, J_o])$$

with $|zI - F_o| = \alpha$. Further, if α is in P_s, then Σ_l is an L-observer for Σ.

For the proof of this theorem, we will need a result which can be easily inferred from a result of Emre [1979, Theorem (2.3) and Remarks (2.7) and (2.8)] which is valid for K = an arbitrary commutative ring with identity.

Lemma 3.5 [7, Theorem 2.3, and Remarks 2.7 and 2.8]: *Let Q be a $p \times p$ nonsingular polynomial matrix such that K_Q is well defined and finite-free. Let R be a $p \times m$ polynomial matrix such that $Q^{-1}R$ is strictly proper. Let Φ be a $p \times r$ polynomial matrix. Then there exist $p \times r$ and $m \times r$ polynomial matrices Q_c and P_c such that*

$$QQ_c + RP_c = \Phi$$

if and only if $Sp_K \pi_Q(\Phi)$ is in the reachable subspace of the Q-realization Σ_Q of $Q^{-1}R$. If this condition holds, P_c can be chosen such that $\delta_{ci}(P_c) \leq$ reachability index of Σ_Q.

Proof of the Theorem 3.4: It is known (see the proof of Theorem 2.4)

how we can construct a column proper polynomial matrix Q'_o with column degrees γ_i, such that $|Q'_o| = \alpha$.

Now with $Q = zI - F'$ and $R = H'$, Lemma 3.5 shows that we can find T', R'_2 with $\delta_{ci}(R'_2) \le \mu - 1$ such that

$$(zI - F')T' + H'R'_2 = L'Q'_o.$$

As Q'_o is column proper with $\delta_{ci}(Q'_o) > \delta_{ci}(R'_2)$, $R'_2 Q'^{-1}_o$ is proper.

$$T' = -(zI - F')^{-1}H'R'_2 + (zI - F')^{-1}L'Q'_o$$

or

$$T'Q'^{-1}_o = -(zI - F')^{-1}H'R'_2Q'^{-1}_o + (zI - F')^{-1}L'$$

i.e., $T'Q'^{-1}_o$ is strictly proper. Then

$$Q_o^{-1}[R_1 : R_2] = Q_o^{-1}[TG : R_2]$$

is proper and $Q_o^{-1}R_1$ is strictly proper. If we choose α to be stable, $|Q_o|$ will be stable. If we choose Σ_L to be the Q_o-realization of $Q_o^{-1}[R_1 : R_2]$, then $|zI - F_o| = \alpha$ (see Remark 2.6 for this property of Q-realizations), and by Lemma 3.3, if α is in P_s, then Σ_L is an L-observer for Σ. □

Remark 3.6: From a practical point of view, choosing $\gamma_i = \mu - 1$ would give us reduced order observers of order $l(\mu - 1)$. Note that we are free to choose γ_i as we want, as long as $\gamma_i \ge \mu - 1$. Note here that as Σ_L is the Q_o-realization of $Q_o^{-1}[R_1 : R_2]$, and Q_o is row proper, the characteristic polynomial of F_o will be α. [See Remarks 2.6 and 2.7.] Hence, we have an L-observer whose characteristic polynomial can be arbitrarily assigned. The constructive methods given in [7] in terms of linear equations over K can be applied to obtain observers (also, regulators) provided that one can solve the corresponding equations over K.

Remark 3.7: For the regulator problem, one obvious choice for L would be I_n, in which case, we will have a full state observer. Then, as the transfer matrix of the composite of Σ and Σ_L is Z_s, one can apply *dynamic state feedback as explained in Section II and construct a* regulator to stabilize (coefficient-assignment) the closed-loop system. For a detailed explanation of the internal stability aspect of the regulators obtained this way, see Remark 3.8.

The advantage of choosing L, other than I_n, would be that, first of all, we would need lower order observers. If there exists an L in $K^{l \times n}$, $l \le n$, such that $L(zI - F)^{-1}G$ can be expressed as $Q^{-1}R$ with row proper Q, then we can construct a lower order compensator for coefficient-assignment of the closed-loop system. Hence, we would reduce the orders of both the observer and the compensator for coefficient-assignment.

In particular, if F' is cyclic with a generator h', we can obtain $h(zI - F)^{-1}G$ using an observer of order $\mu - 1$. Then we can find a monic polynomial q and a polynomial row vector R such that

$$q^{-1}R = h(zI - F)^{-1}G.$$

Then using results of Section II, we can achieve coefficient-assignment with a compensator of order $\nu - 1$ where ν is the reachability index of Σ.

We should note here that if the transfer matrix of $\Sigma = (F, G, H)$ can be expressed as $Q^{-1}R$ with Q, R being polynomial matrices satisfying

$$QA + RB = I$$

for some polynomial matrices A, B which is implied by Σ being split (see [16]) and if in addition such a Q can be chosen to be row proper one can directly use Theorem 2.4 bypassing the need for an observer. This latter condition is equivalent to the fact that the observability indices of (H, F) add up to the dimension of F.

Remark 3.8: We will now show that our technique of regulator synthesis leads to an internally stable closed-loop system. Let $\Sigma = (F, G, H)$ be a finite-free split system. Let $Q_o^{-1}[R_1 : R_2]$ be the transfer matrix of a full state observer as in Theorem 3.4. Then there exists a polynomial matrix T such that

$$T(zI - F) + R_2 H = Q_o.$$

Furthermore, as Q_o is chosen to be row proper, K_{Q_o} is a free K-module. (See Section II.) Let $\Sigma_o = (F_o, [G_o, G_{o2}], H_o, [0 : J_o])$ be a matrix representation of the Q_o-realization of $Q_o^{-1}[T : R_2]$. Notice that since $Q_o^{-1}T$ is strictly proper, there is no nonzero feedthrough term for $Q_o^{-1}T$. As $TG = R_1$, if we define

$$G_{o1} := G_o G,$$

then (F_o, G_{o1}, H_o) is a matrix representation of the Q_o-realization of $Q_o^{-1}R_1$. Thus, $(F_o, [G_{o1}, G_{o2}], H_o, [0 : J_o])$ is a matrix representation of the Q_o-realization of $Q_o^{-1}[R_1 : R_2]$. It can be checked easily (see Remark 2.6) that the characteristic polynomial of F_o is the same as $|Q_o|$. Finally, let $Z_1 = P_1 Q_1^{-1}$ be the compensator which satisfies

$$(zI - F)Q_1 + GP_1 = \Phi$$

where Φ is chosen such that $|\Phi|$ is a stable polynomial. If we choose Φ as in the proof of Theorem 2.4, then by Theorem 2.4 Q_1 is column proper. Then, as explained in Remark 2.6, we can obtain a free realization (F_1, G_1, H_1, J_1) of Z_1 such that the characteristic polynomial of F_1 is $|Q_1|$.

Let x, x_o, and x_1 be the state-variables of Σ, Σ_o, and Σ_1, respectively. Then with a discrete-time interpretation we have the following equations for the closed-loop system, describing its internal behavior:

$$\begin{aligned}
x(t+1) &= Fx(t) + Gu(t),\\
y(t) &= Hx(t),\\
x_o(t+1) &= F_o x_o(t) + G_{o1}u(t) + G_{o2}y(t)\\
y_o(t) &= H_o x_o(t) + J_o y(t)\\
x_1(t+1) &= F_1 x_1(t) + G_1 y_o(t),\\
y_1(t) &= H_1 x_1(t) + J_1 y_o(t),
\end{aligned}$$

and the feedback law

$$u(t) = v(t) - y_1(t)$$

where $v(t)$ is the external input. When $v = 0$, the equations of the overall closed-loop system can be written as

$$\begin{bmatrix} x_1(t+1) \\ x_o(t+1) \\ x(t+1) \end{bmatrix} = \begin{bmatrix} F_1 & G_1 H_o & G_1 J_o H \\ G_{o1} H_1 & F_o + G_{o1} J_1 H_o & G_{o1} J_1 J_o H + G_{o2} H \\ GH_1 & GJ_1 H_o & F + GJ_1 J_o h \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_o(t) \\ x(t) \end{bmatrix}.$$

The closed-loop system is said to be *internally stable* iff

$$\chi := \begin{vmatrix} zI - F_1 & -G_1 H_o & -G_1 J_o H \\ -G_{o1} H_1 & zI - F_o - G_{o1} J_1 H_o & -G_{o1} J_1 J_o H - G_{o2} H \\ -GH_1 & -GJ_1 H_o & zI - F - GJ_1 J_o H \end{vmatrix}$$

is a stable polynomial. Multiplying the third row by G_o and adding to the second row we get

$$\chi = \begin{vmatrix} zI - F_1 & -G_1 H_o & -G_1 J_o H \\ 0 & zI - F_o & -G_{o2} H - G_o(zI - F) \\ -GH_1 & -GJ_1 H_o & zI - F - GJ_1 J_o H \end{vmatrix}.$$

Let us define

$$V := -G_{o2} H - G_o(zI - F).$$

Multiplying the second row by $(zI - F_o)^{-1}$, and further multiplying the second row by $G_1 H_o$ and adding to the first row, we get

$$\chi = \chi_{F_o} \begin{vmatrix} zI - F_1 & 0 & -G_1 J_o H - G_1 H_o (zI - F_o)^{-1} V \\ 0 & I & -(zI - F_o)^{-1} V \\ -GH_1 & -GJ_1 H_o & (zI - F) - GJ_1 J_o H \end{vmatrix}$$

where $\chi_{F_o} := |zI - F_o|$.

Multiply the first row by $(zI - F_1)^{-1}$, and further multiplying the first

row by GH_1, and multiplying the second row by GJ_1H_o and adding both to the third row, we get

$$\chi = \chi_{F_o}\chi_{F_1}\left|\left[(zI - F) - GJ_1J_oH - GH_1(zI - F_1)^{-1}G_1J_oH - GH_1(zI - F_1)^{-1}G_1H_o(zI - F_o)^{-1}V - GJ_1H_o(zI - F_o)^{-1}V\right]\right|.$$

Noting that

$$P_1Q_1^{-1} = H_1(zI - F_1)^{-1}G_1 + J_1$$

and combining terms, we have

$$\begin{aligned}\chi &= \chi_{F_o}\chi_{F_1}\left|\left[(zI - F) - GP_1Q_1^{-1}J_oH - GP_1Q_1^{-1}H_o(zI - F_o)^{-1}V\right]\right| \\ &= \chi_{F_o}\chi_{F_1}\left|\left[(zI - F) - GP_1Q_1^{-1}\left[H_o(zI - F_o)^{-1}V + J_oH\right]\right]\right|.\end{aligned}$$

Now

$$H_o(zI - F_o)^{-1}[G_o : G_{o2}] + [0 : J_o] = Q_o^{-1}[T : R_2]$$

and

$$Q_o^{-1}T(zI - F) + Q_o^{-1}R_2H = I$$

therefore

$$H_o(zI - F_o)^{-1}V + J_oH = I.$$

Thus, we have

$$\begin{aligned}\chi &= \chi_{F_o}\chi_{F_1}|(zI - F) - GP_1Q_1^{-1}| \\ &= \chi_{F_o}\chi_{F_1}|\Phi||Q_1^{-1}|.\end{aligned}$$

As $|Q_1| = \chi_{F_1}$, we have

$$\begin{aligned}\chi &= \chi_{F_o}|\Phi| \\ &= \chi_{F_o}\chi_{F_c}.\end{aligned}$$

Since $\chi_{F_o} = |Q_o|$, and $|\Phi|$ are chosen to be stable polynomials, χ is a stable polynomial. This shows that our technique of regulator synthesis leads to an internally stable system, and hence provides a solution to the regulator problem for finite free split systems over an arbitrary commutative ring. Note that with a nonzero external input v, the transfer matrix from v to x is $Q_1\Phi^{-1}G$ (see Remark 2.6), and Q_o is cancelled. Thus, with our synthesis, regulation and coefficient-assignment are achieved simultaneously.

Remark 3.9: We will now show that strong observability of the pair (F, H) is also a necessary condition for the existence of full state observers with arbitrary characteristic polynomial. Let $\max(K)$ denote the set of all maximal ideals of K. For any m in $\max(K)$, let (F_m, H_m) denote the system over the field K/m obtained by reducing (F, H) modulo m. Then, if the pair (F, H) admits a full state observer with arbitrary characteristic polynomial, it follows that the pair (F_m, H_m) also admits a full state observer with arbitrary characteristic polynomial for each m in $\max(K)$. But as (F_m, H_m) is a system over a field, it is well known that (F_m, H_m) admits a full state observer with arbitrary characteristic polynomial if and only if (F_m, H_m) is observable. Thus, (F_m, H_m) is observable for each m. Now, by dualizing Lemma 3.12 of [23] it follows that (F, H) is strongly observable.

Remark 3.10: It should be noted that observers can be constructed under the weaker hypothesis of detectability [11]. Hence, for detectable and reachable systems, observer and dynamic state-feedback construction can be completed. Further, that such a construction results in an internally stable system is clear from Remark 3.8. Thus, the constructions of [11] can be combined with the techniques developed in Section II to solve the problem of regulation for detectable and reachable systems. The reader is referred to [9] for further generalizations of these ideas to detectable and stabilizable systems. For the connections between these concepts and matrix fraction description over stable rational functions and stable transfer functions, see [17].

IV. Examples

In this section we will consider two examples to illustrate our results on coefficient-assignment, observers, and regulators for linear systems over commutative rings. In the first example, we consider a system (F, G) for which coefficient-assignment can not be achieved by constant state-feedback. We show how by using dynamic state-feedback, coefficient-assignment can be achieved. In the second example, we consider a system (H, F, G) that is not strongly observable. We show how an observer and a regulator can be obtained for this system by using our approach.

Example 4.1: Consider the pair

$$F = \begin{bmatrix} 0 & 0 \\ \sigma & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 \\ 0 & \sigma^2 - 1 \end{bmatrix}$$

over the ring of polynomials $\boldsymbol{R}[\sigma]$ is the indeterminate σ over the field of real numbers. This system is given as an example in [2] of a reachable system that is not coefficient-assignable. In the notation of Corollary 2.5, we have $\nu = 2$. Let us choose $\rho_1 = \rho_2 = 2$, and let $\beta = z^4 + z^2 + 1$. We will obtain a feedback compensator $Z_1 = P_1Q_1^{-1}$ such that the closed-loop system will have β as its characteristic polynomial. It is easily seen that

$$\Phi = \begin{bmatrix} z^2 + 1 & 1 \\ -1 & z^2 \end{bmatrix}. \tag{4.2}$$

Also,

$$\pi_{(zI - F)}(\Phi) = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}. \tag{4.3}$$

If we now solve for P_1 and Q_1 in the polynomial equation

$$(zI - F)Q_1 + GP_1 = \Phi, \tag{4.4}$$

we get

$$P_1 = \begin{bmatrix} \sigma z & 1 \\ -1 & 0 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} z - \sigma & 0 \\ 0 & z \end{bmatrix}. \tag{4.5}$$

Therefore, the transfer matrix z_1 of the feedback compensator is given by

$$Z_1 = P_1Q_1^{-1} = \begin{bmatrix} \sigma z(z - \sigma)^{-1} & z^{-1} \\ -(z - \sigma)^{-1} & 0 \end{bmatrix}. \tag{4.6}$$

With Z_1 as the feedback compensator and

$$Z := (zI - F)^{-1}G \tag{4.7}$$

it can be easily checked the overall closed-loop transfer matrix Z_c is given by

$$\begin{aligned}Z_c &= (I + ZZ_1)^{-1}Z = Q_1\Phi^{-1}G \\ &= \begin{bmatrix} z - \sigma & 0 \\ 0 & z \end{bmatrix}\begin{bmatrix} z^2 + 1 & 1 \\ -1 & z^2 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 \\ 0 & \sigma^2 - 1 \end{bmatrix}.\end{aligned} \tag{4.8}$$

The Q_1' realization of $Q_1'^{-1}P_1$ is given by (F_1', G_1', H_1', J_1') where

$$F_1 = \begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$H_1 = \begin{bmatrix} \sigma^2 & 1 \\ -1 & 0 \end{bmatrix}, \quad J_1 = \begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix}. \tag{4.9}$$

Thus, (F_1, G_1, H_1, J_1) is a free realization of Z_1. The Φ-realization (F_c, G_c, H_c) of the overall closed-loop system transfer matrix Z_c can be easily obtained as explained in Remark 2.6.

A matrix representation of this realization is

$$F_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad G_c = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \sigma^2 - 1 \end{bmatrix}, \quad H_c = \begin{bmatrix} -1 & \sigma & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.10}$$

It can be easily seen that the characteristic polynomial of F_c is $(z^4 + z^2 + 1)$.

Example 4.11: We will now illustrate our technique of observer and regulator synthesis for a delay-differential system. This example has been worked out in detail in [13] and [11, Example 5.6]. It will be seen that our observer turns out to be the same as that obtained in [11 Example 5.6]. We then complete the regulator synthesis by obtaining a feedback compensator to stabilze the system.

In this example, the system is not strongly observable. This example shows that observer synthesis can be approached with our results even if the system is not strongly observable. The reader is referred to [11] for a generalization of the concept of detectability for algebras over fields.

The delay-differential system is given by

$$\begin{aligned} \dot{x}_1 &= x_2(t-1) + u(t),\\ \dot{x}_2 &= x_1(t-1) + x_2(t) + u(t),\\ y(t) &= x_2(t). \end{aligned} \tag{4.12}$$

Let σ denote the shift operator $\sigma(x(t)) := x(t-1)$. Then the delay-differential system (4.12) can be modeled as a linear system over $\boldsymbol{R}[\sigma]$. (For details, see [14].) The matrices (F, G, H) are given by

$$F = \begin{bmatrix} 0 & \sigma \\ \sigma & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad H = [0 \quad 1].$$

It is shown in [11] that this system is not strongly observable and is unstable. It is also shown that the polynomial $z + \mu\sigma$ is stable for $0 \leq \mu \leq \pi/2$. We will construct a full state observer whose characteristic polynomial is $z + \mu\sigma$. With the notation of Section III, we need to solve for T' and R'_2 in the equation

$$(zI - F')T' + H'R'_2 = Q'_o. \tag{4.14}$$

Let us choose

$$Q'_o = \begin{bmatrix} z + \mu\sigma & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.15}$$

Now a solution to (4.14) is given by

$$R_2 = \begin{bmatrix} \mu z + \sigma - \mu \\ 1 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 0 \\ -\mu & 0 \end{bmatrix}. \tag{4.16}$$

Furthermore,

$$R_1 = TG = \begin{bmatrix} 1 - \mu\sigma - \mu \\ 0 \end{bmatrix}.$$

Therefore, the observer transfer matrix is given by

$$\begin{aligned} &Q_o^{-1}[R_1 : R_2] \\ &= \begin{bmatrix} (1-\mu\sigma-\mu)(z+\mu\sigma)^{-1} & (\sigma-\mu-\mu^2\sigma)(z+\mu\sigma)^{-1} + \mu \\ 0 & 1 \end{bmatrix}. \end{aligned} \tag{4.17}$$

The following observer equations can be immediately obtained from (4.17) as explained in Section III:

$$\begin{aligned} \dot{\hat{x}}(t) &= -\mu\hat{x}(t-1) + (1-\mu)u(t) - \mu u(t-1) \\ &\quad + (1-\mu^2)y(t-1) - \mu y(t), \\ \hat{y}_1(t) &= \hat{x}(t) + \mu y(t), \\ \hat{y}_2(t) &= y(t). \end{aligned} \tag{4.18}$$

If μ is chosen such that $0 \leq \mu \leq \pi/2$, then the system (4.18) is a full state observer for the system (4.12). It is thus seen that observer synthesis is possible even if the original system is not strongly observable. The key point is to choose Q_o such that $|Q_o|$ is stable, and such that solutions to (4.14) exist.

We will now construct a feedback compensator to stablize the overall system. Using the notations of Section II, we have $\nu = 2$. Let us choose $\rho_1 = \rho_2 = 2$. Let

$$\beta := z^4 + 2z^3 + 3z^2 + 2z + 1 = (z^2 + z + 1)^2. \tag{4.19}$$

It can be easily checked that β is a stable polynomial for continuous time systems. The polynomial matrix Φ turns out to be

$$\Phi = \begin{bmatrix} z^2 + 2z + 3 & 2z + 1 \\ -1 & z^2 \end{bmatrix}. \tag{4.20}$$

We need to solve for P_1 and Q_1 in the polynomial equation

$$(zI - F)Q_1 + GP_1 = \Phi. \tag{4.21}$$

It is easy to check that

$$\pi_{(zI-F)}(\Phi) = \begin{bmatrix} \sigma^2 + 3 & \sigma + 1 \\ 3\sigma - 1 & \sigma^2 + 2\sigma + 1 \end{bmatrix}.$$

A solution for P_1 and Q_1 is given by

$$\begin{aligned} P_1 = \big[&(3\sigma - \sigma^2 - 4)z + (\sigma^3 - 2\sigma^2 + 4\sigma + 3) : \\ &(\sigma^2 - \sigma)z + (1 + \sigma - \sigma^2 - \sigma^3)\big], \end{aligned}$$

$$Q_1 = \begin{bmatrix} z + (\sigma - 3\sigma + \sigma^2) & 2 - \sigma - \sigma^2 \\ \sigma^2 - 2\sigma + 4 & z + (1 - \sigma - \sigma^2) \end{bmatrix}. \tag{4.22}$$

Now the transfer matrix of the feedback compensator is given by $P_1Q_1^{-1}$, with P_1, Q_1 as in (4.22). The Q_1-realization for the compensator can be obtained as explained in Remark 2.6.

The overall transfer matrix with the observer and feedback compensator is given by

$$Z_c = HQ_1\Phi^{-1}G. \tag{4.23}$$

The Φ-realization for (4.23) is given by the following equations:

$$\begin{aligned} \dot{\tilde{x}}_1(t) &= \tilde{x}_2(t), \\ \dot{\tilde{x}}_2(t) &= \tilde{x}_3(t) + u(t), \\ \dot{\tilde{x}}_3(t) &= \tilde{x}_4(t) \\ \dot{\tilde{x}}_4(t) &= -\tilde{x}_1(t) - 2\tilde{x}_2(t) - 3\tilde{x}_3(t) - 2\tilde{x}_4(t) + u(t), \\ y(t) &= \tilde{x}_1(t) - \tilde{x}_1(t-1) - \tilde{x}_1(t-2) + \tilde{x}_2(t) + 4\tilde{x}_3(t) \\ &\quad - 2\tilde{x}_3(t-1) + \tilde{x}_3(t-2). \end{aligned}$$

Acknowledgment

The authors wish to thank Prof. R. E. Kalman for several discussions on the subject studied in this paper. The authors are also grateful to the reviewers for their useful suggestions.

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